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COMBINATIONAL COMBINATIONS.

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BY

MOROMOTO NISHI,

TOKYO, JAPAN.

1900.

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PREFACE.

Algebra treats, in most parts, how to combine the symbols of quantities; and the theory of combination especially deals with certain laws of combining several things or numbers.

In this pamphlet, I shall explain some principles relating to complicate cases of combinations premising certain truth as acquainted.

MOROMOTO NISHI.

Tokyo, January 27th, 1900.

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Chapter I. Combinational Theorem.

Chapter II. Figurate numbers in determinant.

CHAPTER I.

COMBINATIONAL THEOREM.



Additional Corrigenda.

Read,	for,	page,	line.
nC_r,	O_n	3,	4.
$+{}^{n+r-3}C_{r-1} + {}^{n+r-2}C_{r-1},$	$+{}^{n-2}C_{r-1} + {}^{n-1}C_{r-1},$	7,	21.

of x and y , it must be true for all values of the same variables.

2. Substitute $-x$ and $-y$ for x and y in the formula (A), conceive to change the order of consecutive factors in each of permutational factors, and divide the both sides of the equality by $(-1)^{n+1}$; then

CONTENTS.

Chapter I. Combinational Theorem.



CHAPTER I.

COMBINATIONAL THEOREM.



1. Expand the identity

$$(1-z)^{-(x+y)} \equiv (1-z)^{-x}(1-z)^{-y}$$

and equate the coefficients of z^{n+1} in the both sides; then we have

$$\begin{aligned} {}^{x+y}H_{n+1} = & {}^xH_{n+1} + {}^xH_n {}^yH_1 + {}^xH_{n-1} {}^yH_2 + \dots \\ & + {}^xH_2 {}^yH_{n-1} + {}^xH_1 {}^yH_n + {}^yH_{n+1}. \end{aligned}$$

Multiply each side of this equality by $|n+1$; then

$$\left(\text{since } {}^xH_r = \frac{(x+r-1)_r}{|r|}, \right)$$

$$\begin{aligned} (x+y+n)_{n+1} = & (x+n)_{n+1} + {}^{n+1}C_1(x+n-1)_n y_1 \\ & + {}^{n+1}C_2(x+n-2)_{n-1}(y+1)_2 + \dots \\ & + {}^{n+1}C_{n-1}(x+1)_2(y+n-2)_{n-1} + {}^{n+1}C_n x_1(y+n-1)_n \\ & + (y+n)_{n+1} \quad (A). \end{aligned}$$

Since the formula (A) is true for more than $n+1$ values of x and y , it must be true for all values of the same variables.

2. Substitute $-x$ and $-y$ for x and y in the formula (A), conceive to change the order of consecutive factors in each of permutational factors, and divide the both sides of the equality by $(-1)^{n+1}$; then

$$(x+y)_{n+1} = x_{n+1} + {}^{n+1}C_1 x_n y_1 + {}^{n+1}C_2 x_{n-1} y_2 + \dots \\ + {}^{n+1}C_{n-1} x_2 y_{n-1} + {}^{n+1}C_n x_1 y_n + y_{n+1}.$$

Hence the formula (A) is the converse of Vandermonde's Theorem; so that each proves the truth of other for negative values of x and y .

3. The whole number of ways of selecting x things r at a time is known to be ${}^x C_r$.

Suppose x things are divided into n groups, there being y, z, w , etc. things respectively; then r things can be taken, in all possible ways, out of one or more of the n groups.

Now, the number of taking r things out of each group separately is clearly $\Sigma {}^y C_r$ altogether.

Also, the number of taking r things out of every two groups is $\Sigma {}^y C_{r-s} {}^s C_r$ for any particular integral values of r and s ; therefore, if we adopt a notation $N[\Sigma {}^y C_{r-s} {}^s C_r]$ for the sum

$$\Sigma {}^y C_{r-1} {}^1 C_1 + \Sigma {}^y C_{r-2} {}^2 C_2 + \dots + \Sigma {}^y C_{r-s} {}^s C_s + \dots,$$

provided $s \leq r-s$, the total number of taking r things out of every two groups may be denoted by the sum $N[\Sigma {}^y C_{r-s} {}^s C_s]$.

Similarly, since the number of taking out of every three groups is $\Sigma {}^y C_{r-s} {}^s C_{s-t} {}^t C_t$ for any particular integral values of r, s and t ; the total number for all possible values of r, s and t , provided $t \leq s-t \leq r-s$, may be denoted by $N[\Sigma {}^y C_{r-s} {}^s C_{s-t} {}^t C_t]$.

And, in general, the total number of taking r things out of every k groups may be denoted by

$$N[\Sigma ({}^y C_{r-s} {}^s C_{s-t} {}^t C_{t-p} \dots)], \text{ provided} \\ r-s \leq s-t \leq t-p \leq \dots,$$

each term of the sum consisting of k combinational factors.

Hence, we have the following

Formula.

$$C_r \equiv \Sigma^r C_r + N[\Sigma^r C_{r-s} {}^s C_s] + N[\Sigma^r C_{r-s} {}^s C_{s-t} {}^t C_t] \\ + \dots \dots \dots + N[\Sigma({}^r C_{r-s} {}^s C_{s-t} {}^t C_{t-p} \dots \dots \dots)] + \\ \dots \dots \dots (B),$$

where $x = y + z + w + \dots \dots \dots$, and

$$r - s \geq s - t \geq t - p \geq \dots \dots \dots.$$

Multiply each side of the formula (B) by $|r$; then

$$(y + z + w + \dots \dots \dots)_r \equiv \\ \Sigma y_r + N[{}^r C_s \Sigma y_{r-s} z_s] + N[{}^r C_s {}^s C_t \Sigma y_{r-s-z} w_t] \\ + \dots \dots \dots + N[({}^r C_s {}^s C_t {}^t C_p \dots \dots \dots) \Sigma(y_{r-s-z-t-w-p} \dots \dots \dots)] \\ + \dots \dots \dots (B').$$

This is true for all values of y, z, w , etc.

Example. When $n=2$, the formula (B') becomes

$$(y+z)_r \equiv \Sigma y_r + N[{}^r C_s \Sigma y_{r-s} z_s]$$

This is the contracted form of Vandermonde's Theorem.

4. By means of the formula (B'), the Binomial Theorem

$$(1+x)^m (1+x)^n (1+x)^l \dots \dots \dots = (1+x)^{m+n+l+\dots \dots \dots}$$

can be directly proved for all values of m, n, l , etc.

Conversely, we can prove the truth of (B') as follows.

Consider the product

$$(1-x)^{-p} (1-x)^{-q} (1-x)^{-r} \dots \dots \dots = \\ (1 + {}^p H_1 x + {}^p H_2 x^2 + \dots \dots \dots + {}^p H_r x^r + \dots \dots \dots) \times \\ (1 + {}^q H_1 x + {}^q H_2 x^2 + \dots \dots \dots + {}^q H_r x^r + \dots \dots \dots) \times \\ (1 + {}^r H_1 x + {}^r H_2 x^2 + \dots \dots \dots + {}^r H_r x^r + \dots \dots \dots) \times \\ \dots \dots \dots ;$$

the coefficients of x^{r+1} in both sides can be seen to make up the equality

$$\begin{aligned} (v+z+w+\dots)H_{r+1} = \\ \Sigma^v H_{r+1} + N[\Sigma^v H_{r-s+1} {}^s H_s] + N[\Sigma^v H_{r-s+1} {}^s H_{s-t} {}^t H_t] \\ + \dots \dots \dots + N[\Sigma^v H_{r-s+1} {}^s H_{s-t} {}^t H_{t-p} \dots] + \\ \dots \dots \dots (C), \end{aligned}$$

provided $r-s+1 \leq s-t \leq t-p \leq \dots$.

Multiply each side of (C) by $|r+1|$; then

$$\begin{aligned} (y+z+w+\dots+r)_{r+1} = \\ \Sigma(y+r)_{r+1} + N[{}^{r+1}C_s \Sigma(y+r-s)_{r-s+1} (z+s-1)_s] \\ + N[{}^{r+1}C_s {}^s C_t \Sigma(y+r-s)_{r-s+1} (z+s-t-1)_{s-t} (w+t-1)_t] \\ + \dots \dots \dots + \\ N[({}^{r+1}C_s {}^s C_t {}^t C_p \dots) \Sigma\{(y+r-s)_{r-s+1} (z+s-t-1)_{s-t} \\ (w+t-p-1)_{t-p} \dots\}] + \dots \dots \dots (C') \end{aligned}$$

In (C'), replace y, z, w , etc. by $-y, -z, -w$, etc., and conceive to change the order of consecutive factors in each of permutational factors; then

$$\begin{aligned} (-1)^{r+1} (y+z+w+\dots)_{r+1} = \\ (-1)^{r+1} \{ \Sigma y_{r+1} + N[{}^{r+1}C_s \Sigma y_{r-s+1} z_s] + \dots \dots \dots \\ + N[{}^{r+1}C_s {}^s C_t {}^t C_p \dots y_{r-s+1} z_{s-t} w_{t-p} \dots] \\ + \dots \dots \dots \} \end{aligned}$$

This proves the truth of the combinational theorems [(B') and (C')] for opposite signs; because, in $\Pi(1-x)^{-v}$ and $\Pi(1-x)^v$, the coefficients of x^{r+1} are clearly $(v+z+w+\dots)H_{r+1}$ and $(-1)^{r+1} (v+z+w+\dots)C_{r+1}$.

CHAPTER II.

FIGURATE NUMBERS IN DETERMINANT.

5. The series of figurate numbers

1	1	1	1	1
1	2	3	4	5	.	.	.	n
1	3	6	10	15
1	4	10	20	35
1	5	15	35	70
.
.
.
.	n	${}^nH_{n-1}$

since

$$1 + {}^{n-1}H_1 + {}^{n-1}H_2 + \dots + {}^{n-1}H_{r-1} + {}^{n-1}H_r = {}^nH_r,$$

may be written as

1	1	1	1	1	.	.	.	1
1	2H_1	2H_2	2H_3	2H_4	.	.	.	${}^2H_{n-1}$
1	3H_1	3H_2	3H_3	3H_4	.	.	.	${}^3H_{n-1}$
1	4H_1	4H_2	4H_3	4H_4	.	.	.	${}^4H_{n-1}$
1	5H_1	5H_2	5H_3	5H_4	.	.	.	${}^5H_{n-1}$
.
.
.
1	nH_1	nH_2	nH_3	nH_4	.	.	.	${}^nH_{n-1} (D).$

$$\begin{aligned} &(1+x)^0, \\ &(1+x)^1, \\ &(1+x)^2, \\ &(1+x)^3, \\ &\dots\dots\dots, \\ &(1+x)^r. \end{aligned}$$

Hence, the table (*D*) is equivalent to

1	1	1	1	1	.	.	.	1
1	2C_1	3C_2	4C_3	.	.	.	${}^rC_{r-1}$.
1	3C_1	4C_2	.	.	.	${}^rC_{r-2}$.	.
1	4C_1
1
.
.	.	rC_2
.	rC_1
1	${}^{2r}C_r$	(<i>D''</i>).

7. The following relations are plain from the tables (*D*), (*D'*) and (*D''*):

- (1) ${}^nH_r = 1 + {}^rH_1 + {}^rH_2 + \dots\dots\dots + {}^rH_{n-2} + {}^rH_{n-1},$
- (2) ${}^nH_r = 1 + {}^{n-1}C_1 + {}^nC_2 + \dots\dots\dots + {}^{n+r-2}C_{r-1} + {}^{n+r-2}C_r,$
- (3) ${}^nH_r = 1 + {}^rC_{r-1} + {}^{r+1}C_{r-1} + \dots\dots\dots + {}^{n-2}C_{r-1} + {}^{n-1}C_{r-1}.$

8. The figurate numbers in determinant.

The determinant in which the successive rows are the series of figurate numbers as

U of M

$$\begin{vmatrix} 1 & 1 & 1 & 1 & . & . & . \\ 1 & {}^2H_1 & {}^2H_2 & {}^2H_3 & . & . & . \\ 1 & {}^3H_1 & {}^3H_2 & {}^3H_3 & . & . & . \\ 1 & {}^4H_1 & {}^4H_2 & {}^4H_3 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{vmatrix} = |F_{\infty}|,$$

represents the elimination of the successive integral powers of $\frac{1}{1-x}$ to infinity.

To prove $|F_{\infty}| = 1$.

Assume

$$\frac{1}{1-x} = X = 0,$$

$$\left(\frac{1}{1-x}\right)^2 = X^2 = 0,$$

$$\left(\frac{1}{1-x}\right)^3 = X^3 = 0,$$

etc.,

and introduce a complemental X^0 for convenience ; then the elimination of X^0, X, X^2, X^3 , etc. may be denoted by

$$\begin{vmatrix} 1 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 0 & . & . & . \\ 0 & 0 & 0 & 1 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 \end{vmatrix} = 1.$$

Q. E. D.

If the first row and column for $X^0=1$ be suppressed, still the determinant will reserve the similar form.

Hence

$$|F^\infty| = |1| = 1.$$

Theorem. *The value of figurate numbers in determinant is always equivalent to unity.*

To prove $|F_4| = 1$.

By modifying the notation in (D')

$$|F_4| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & {}^2C_1 & {}^3C_1 & {}^4C_1 \\ 1 & {}^3C_1 & {}^4C_2 & {}^5C_2 \\ 1 & {}^4C_1 & {}^5C_2 & {}^6C_3 \end{vmatrix};$$

by subtracting all the elements of each row from the corresponding elements of next row (since ${}^nC_r - {}^{n-1}C_r = {}^{n-1}C_{r-1}$)

$$|F_4| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & {}^2C_1 & {}^3C_1 \\ 0 & 1 & {}^3C_2 & {}^4C_2 \\ 0 & 1 & {}^4C_3 & {}^5C_3 \end{vmatrix} = \begin{vmatrix} 1 & {}^2C_1 & {}^3C_1 \\ 1 & {}^3C_2 & {}^4C_2 \\ 1 & {}^4C_3 & {}^5C_3 \end{vmatrix};$$

by subtracting all the elements of each column from the corresponding elements of next column,

$$\begin{vmatrix} 1 & {}^2C_1 & {}^3C_1 \\ 1 & {}^3C_2 & {}^4C_2 \\ 1 & {}^4C_3 & {}^5C_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & {}^2C_1 & {}^3C_1 \\ 1 & {}^3C_1 & {}^4C_2 \end{vmatrix} \quad (\text{since } {}^3C_2 = {}^3C_1 \text{ and } {}^4C_3 = {}^4C_2)$$

U of M

$$= |F_3|.$$

Hence

$$|F_4| = |F_3|.$$

Similarly

$$|F_3| = |F_2|;$$

and

$$|F_2| = \begin{vmatrix} 1 & 1 \\ 1 & {}^2C_1 \end{vmatrix} = 1.$$

Therefore

$$|F_4| = 1.$$

This proof is perfectly general.